Frequency domain analysis method of nonstationary random vibration based on evolutionary spectral representation

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Abstract

A novel frequency domain approach, which combines the pseudo excitation method modified by the authors and multi-domain Fourier transform (PEM-FT), is proposed for analysing nonstationary random vibration in this paper. For a structure subjected to a nonstationary random excitation, the closed-form solution of evolutionary power spectral density (EPSD) of the response is derived. Because the deterministic process and random process in an evolutionary spectrum are separated effectively using this method during the analysis of nonstationary random vibration of a linear damped system, only the response's modulation function of the system needs to be estimated, which brings about a large saving in computational time. The method is general and highly flexible since it can deal with various damping types and nonstationary random excitations with different modulation functions. In the numerical examples, nonstationary random vibration of several distinct structures (a truss subjected to base excitation, a mass-spring system with exponential damping, a beam on a Kelvin foundation under a moving random force and a cable-stayed bridge subjected to earthquake excitation) is studied. The results obtained by the PEM-FT are compared with other methods and show its validity and superior effectiveness.

Keywords: nonstationarity; random vibration; evolutionary spectrum; pseudo excitation method; frequency domain method

1 Introduction

Some environmental loads, such as earthquakes, wind gusts, etc., which must be considered in engineering structural design, possess intrinsic nonstationary random characteristics as their probability distributions vary with time [1]. The modelling of nonstationary excitations has been an important subject of study for a long time, for instance Gabor analysis [2], double frequency spectrum [3], evolutionary spectrum [4], time scale model with wavelet transform [5], method based on polynomial chaos expansion [6], smooth decomposition method [7], and polynomial-algebraic method [8] are developed to characterize nonstationary random processes. The spectral method has great advantages in representation of nonstationary random processes due to its unique form of energy distribution corresponding to frequencies. In modelling ground motion in earthquake engineering, waves in offshore engineering, road roughness in vehicle engineering, the spectral representation method is widely adopted [9-11].

The spectral structure of nonstationary random processes includes double-frequency spectral model and frequency-time spectral model. The concept of double-frequency spectrum is extended from single spectral representation of a stationary process and the double frequencies are employed to represent the statistical characteristics. For nonstationary random vibration analysis of a linear time-invariant / time variant system, the double-frequency spectral responses of the system can be found using matrix product operation between the double-frequency spectrum of excitation and the frequency response function of the system in one generic and compact formula [3]. But the applications of double-frequency spectral method are limited since its physical meaning is not easy to explain, and more importantly double-frequency spectrums of input loads are rarely available in reality at present [12]. To overcome the shortcoming of double-frequency spectrum method, other methods have been developed, for instance, Sun and Greenberg [13] introduced the follow-up spectral analysis procedure to deal with the dynamic response of linear systems induced by a moving load. Another way of representing the nonstationary random process is evolutionary spectrum, which can model uniformity/non-uniformity modulated evolutionary random excitations. In earthquake engineering the evolutionary spectrum is applied widely due to the clear physical meaning as instantaneous power spectral density [14-15]. Generally speaking, for nonstationary random vibration of a system subjected to uniformity/non-uniformity modulated random excitations, time domain analysis or frequency-time analysis is always required. Iwan and Mason [16] derived an ordinary differential equation for the covariance matrix and studied the response of nonlinear discrete systems under a nonstationary random excitation based on an evolutionary equation of statistical moments. Sun and Kareem [17] investigated the dynamic response of a multi-degree-of-freedom system subjected to a nonstationary colour vector-valued random excitation in the time-domain. Lin, et al. [18] presented an algorithm for nonstationary responses of structures under evolutionary random seismic excitations in the frequency-time domain. Di Paola and Elishakoff [19] investigated the non-stationary response of linear systems subjected to normal and non-normal generally non-stationary excitations. Smyth and Masri [20] proposed a new method based on equivalent linearization approaches for estimating the nonstationary response of nonlinear systems under nonstationary excitation process. Duval, et al [21] studied nonzero mean Root-Mean-Square (RMS) response of a SDOF system with a shape memory restoring force. In the investigations of closed-form solutions for the response of linear systems to nonstationary excitation, Conte and Peng [22-23] introduced explicit, closed-form solutions for the correlation matrix and evolutionary power spectral density matrix of the response of a system under uniformly modulated random process. Muscolino and Alderucci [24-25] established a method to evaluate the closed-form solution of the evolutionary power spectral response of classically damped linear structural systems subjected to both separable and non-separable nonstationary excitations.

Based on frequency-time analysis strategy, the random vibration of a structure with viscous damping subjected to nonstationary random excitation has been extensively investigated in the afore-mentioned references. However, anon-viscous damping model is appropriate to describe damping characteristics of composite materials because it takes into account the dependence of damping on stresses and displacements and provides a better representation of energy dissipation of real materials undergoing forced vibration. But, if anon-viscous damping model is adopted, such as exponential damping model, complex-valued stiffness, etc., the numerical integration used in the dynamic analysis of structures has been known to become unstable. Some researchers pointed out that the main reason for the unstable numerical integration was the presence of unstable poles of the equation of motion of the structure and hence a special operation must be executed in the state space to achieve a stable analysis in the time domain [26-27]. For such problems, frequency domain method provides an alternative analysis scheme, for example, Pan and Wang [28] introduced the DFT/FFT method to exponentially damped linear systems subjected to arbitrary initial conditions and evaluated the accuracy by comparing the results with those obtained from the state-space method in the time-domain. On the other hand, there is a wider class of problems, in which the equations of motion are formulated in terms of the dynamic stiffness matrix in the frequency domain, for instance, the transfer matrix adopted in wave propagation problems [29] and the complex frequency-dependent damping models [30]. There are not many investigations on the random vibration analysis of systems with various types of damping due to the numerical difficulty mentioned above. Only stationary random vibration analysis has been reported, for example, Dai, et al. [31] studied the responses of laminated composite structures attached with a frequency-dependent damping layer subjected to a stationary random excitation. In addition, for frequency-time analysis of nonstationary random vibration, a step-by-step integration must be executed at each of the frequencies involved, when the numerical method is used. In order to predict accurately responses of a system a small time step must be adopted when the excitations contain high-frequency components and thus such an analysis incurs high computational cost. To develop efficient and accurate nonstationary random vibration analysis algorithms to overcome the shortcomings of traditional frequency-time methods is the major motivation of this paper.

For a linear time-invariant systems, the pseudo excitation method, which transforms stationary random vibration analysis into harmonic vibration analysis and nonstationary random vibration analysis into deterministic time domain analysis, has been widely used in several fields [32-34]. It is worth noting that for broadband random vibration with frequency modulation De Rosa et al. gave another derivation of the classical PEM. They developed a pseudo-equivalent deterministic excitation method (PEDEM) for a complex structure with a turbulent boundary layer, and their results showed that the PEDEM had good accuracy and greatly reduced the computational cost [35].

The traditional pseudo excitation method (TPEM) for nonstationary random vibration analysis belongs to a category of frequency-time method. To achieve a good compromise between computational accuracy and efficiency, the precise integration method is recommended for time domain analysis at each frequency point. Depending on the modulation function of the system, the appropriate integration form can be selected [32]. In this paper, the pseudo excitation method is improved to establish the input-output relationship in the frequency domain and then the nonstationary random vibration analysis is transformed into modulation function analysis of the output, which leads to excellent accuracy and high efficiency. The structure of the paper is organized as follows: In section 2, the evolutionary power spectral density (EPSD) model of the nonstationary random process is given and it is expressed by a product function of the deterministic modulation term and a stationary random term. In section 3, the PEM and FT are developed for nonstationary vibration analysis and the closed-form solution of EPSD of the system responses is derived. The discrete Fourier transformation (DFT) is investigated in the context of nonstationary random vibration. In section 4, the application of the frequency domain analysis method of nonstationary random vibration is investigated for proportionally damped systems, complex-value damped systems and exponentially damped systems, respectively. In addition, by introducing a concept of ‘equivalent modulation function’, the proposed method can also be used for the nonstationary random vibration analysis of an infinite beam resting on a Kelvin foundation under a moving random load. In section 5, the nonstationary random vibration analysis of a truss structure, a mass-spring system with exponential damping, a beam on Kelvin foundation under a moving random force and a cable-stayed bridge is carried out, respectively. The numerical results obtained by the proposed method are compared with the frequency-time method to show the effectiveness and advantages of the proposed method. In section 6, the features of proposed frequency domain method to nonstationary random vibration analysis are summarized briefly.

2 EPSD description of a nonstationary random process

Using EPSD, a non-stationary process can be defined by the Fourier-Stieltjes integration below [36]

$$x(t) = \int_{-\infty}^{+\infty} \exp(\mathrm{i}\omega t)\, a(\omega, t)\mathrm{d}Z(\omega) \tag{1}$$

in which $Z(\omega)$ is an orthogonal incremental process whose increments $\mathrm{d}Z(\omega_1)$ and $\mathrm{d}Z(\omega_2)$ at any two different frequency points ω_1 and ω_2 are uncorrelated random variables, and satisfy the following conditions

$$\mathrm{E}[\mathrm{d}Z(\omega_1)\mathrm{d}Z^*(\omega_2)] = \delta(\omega_1 - \omega_2)S_{uu}(\omega_1)\mathrm{d}\omega_1\mathrm{d}\omega_2 \tag{2}$$

where superscript ‘*’ denotes complex conjugate, $\delta(\cdot)$ is Dirac delta function, $\mathrm{E}[\blacksquare]$ is expectation operator, $S_{uu}(\omega)$ is power spectral density, it is a real symmetric function that has the property of $S_{uu}(-\omega) = S_{uu}(\omega)$.

In Eq. (1) $a(\omega, t)$ is the slowly varying deterministic time-frequency modulation function. If $a(\omega, t)$ is a constant, $x(t)$ converted reduces into a stationary random process; if $a(\omega, t) \equiv$

$a(t)$, $x(t)$ converted reduces into a uniformly modulated nonstationary random process. Compared with the non-uniform modulation process, the uniform modulation process is used more widely. There are several types of modulation functions available, such as the segmentation function type, the exponential function type and the combination type, etc. [37-38]. A uniform modulation nonstationary process is used to describe the excitation load in this paper.

A uniformly modulated nonstationary random process $x(t)$ is an oscillatory process with an EPSD, which is defined by an oscillatory function family $\mathcal{F}_k = \{\mathrm{e}^{\mathrm{i}\omega t} a_k(t)\}$. Using Eqs. (1) and (2), the uniformly modulated nonstationary random process $x(t)$ can be expressed by the autocorrelation function in the following form [24]

$$R_{xx}(t_1, t_2) = \int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}\omega(t_2 - t_1)} a^*(t_1) a(t_2) S_{uu}(\omega) \mathrm{d}\omega = \int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}\omega(t_2 - t_1)} S_{xx}(\omega, t_1, t_2) \mathrm{d}\omega \quad (3)$$

The evolutionary power spectrum of the random process $x(t)$ can be expressed as

$$S_{xx}(\omega, t) = |a(t)|^2 S_{uu}(\omega) \quad (4)$$

in which $|\blacksquare|$ denotes the norm of a function. Eq. (4) is called the evolutionary power spectral density function for the nonstationary random process $x(t)$.

3 Frequency domain method of nonstationary random vibration analysis

3.1 Closed-form solution of EPSD of nonstationary random vibration responses

Uniform evolutionary random process $x(t)$ is a nonstationary random process model adopted widely in engineering and its mathematical expression is given by [38]

$$x(t) = a(t) u(t) \quad (5)$$

where $a(t)$ is a slowly varying envelope function; $u(t)$ is a stationary random process with zero mean value, its auto power spectral density $S_{uu}(\omega)$ and auto correlation function $R_{uu}(t)$ are known.

Considering a nonstationary random process $x(t)$ acting as the input to a time-invariant linear system, the responses of the system, denoted as output $\mathbf{y}(t)$, can be obtained by the Duhamel integral as

$$\mathbf{y}(t) = \int_{-\infty}^{+\infty} \mathbf{h}(\tau) \mathbf{e} x(t - \tau) \mathrm{d}\tau \quad (6)$$

where $\mathbf{e}$ is the index vector of excitation whose elements are either 1 or zero; $\mathbf{h}(\tau)$ is impulse response function matrix of the system. There is a well-known relationship between the Fourier transform of the impulse response function matrix $\mathbf{h}(\tau)$ and the frequency response function matrix $\mathbf{H}(\omega)$ of the system [39]

$$\mathbf{H}(\omega) = \int_{-\infty}^{\infty} \mathbf{h}(\tau)\mathrm{e}^{-\mathrm{i}\omega\tau}\mathrm{d}\tau,\ \ \mathbf{h}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbf{H}(\omega)\mathrm{e}^{\mathrm{i}\omega\tau}\mathrm{d}\omega \tag{7}$$

The product of responses at two time instants t_1 and t_2 can be expressed as

$$\begin{aligned}\mathbf{y}(t_1)\mathbf{y}^{\mathrm{T}}(t_2) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mathbf{h}(\tau_1)\mathbf{e}\mathbf{e}^{\mathrm{T}}\mathbf{h}^{\mathrm{T}}(\tau_2)a(t_1-\tau_1)\,u(t_1-\tau_1)\,\cdot \\ a(t_2-\tau_2)u(t_2-\tau_2)\mathrm{d}\tau_1\mathrm{d}\tau_2\end{aligned} \tag{8}$$

where ‘T’ denotes matrix or vector transposition.

Furthermore, the response autocorrelation matrix can be expressed as [3]

$$\begin{aligned}\mathbf{R}_{yy}(t_1,t_2) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mathbf{h}(\tau_1)\mathbf{e}\mathbf{e}^{\mathrm{T}}\mathbf{h}^{\mathrm{T}}(\tau_2)a(t_1-\tau_1)a(t_2-\tau_2)\,\cdot \\ E[u(t_1-\tau_1)u(t_2-\tau_2)]\mathrm{d}\tau_1\mathrm{d}\tau_2\end{aligned} \tag{9}$$

Since $u(t)$ is a stationary random process, the relation between auto correlation function $R_{uu}(\tau)$ and auto power spectral density $S_{uu}(\omega)$ can be given by [1]

$$E[u(t_1-\tau_1)u(t_2-\tau_2)] = R_{uu}(\tau) = \int_{-\infty}^{+\infty} S_{uu}(\omega)\mathrm{e}^{\mathrm{i}\omega\tau}\mathrm{d}\omega \tag{10}$$

where $\tau = (t_2-\tau_2)-(t_1-\tau_1)$, and the Wiener - Khintchine relation has been used.

Substitution of Eq. (10) into Eq. (9) leads to

$$\mathbf{R}_{yy}(t_1,t_2) = \int_{-\infty}^{+\infty} \widetilde{\mathbf{\Theta}}^{*}(\omega,t_1)\widetilde{\mathbf{\Theta}}^{\mathrm{T}}(\omega,t_2)\mathrm{d}\omega \tag{11}$$

where $\widetilde{\mathbf{\Theta}}$ is given by

$$\widetilde{\mathbf{\Theta}}(\omega,t) = \int_{-\infty}^{+\infty} \mathbf{h}(\tau)\mathbf{e}\tilde{x}(t-\tau,\omega)\mathrm{d}\tau \tag{12}$$

$$\tilde{x}(t,\omega) = a(t)\sqrt{S_{uu}(\omega)}\mathrm{e}^{\mathrm{i}\omega t} \tag{13}$$

where, $\tilde{x}(t,\omega)$ is called pseudo excitation [18, 32] and $\widetilde{\mathbf{\Theta}}(\omega,t)$ is the pseudo response vector of the system subjected to pseudo excitation $\tilde{x}(t,\omega)$.

Taking ω is as a parameter that does not take part in the following integration, the Fourier integral transform of Eq. (13) with respect to time t is executed and the frequency domain expression of pseudo excitation $\tilde{x}(t,\omega)$ can be given by

$$\tilde{X}(\theta,\omega) = \int_{-\infty}^{+\infty} \tilde{x}(t,\omega)\mathrm{e}^{-\mathrm{i}\theta t}\mathrm{d}t = \sqrt{S_{uu}(\omega)}A(\theta-\omega) \tag{14}$$

The corresponding inverse Fourier transform can be written as

$$\tilde{x}(t,\omega)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\tilde{X}(\theta,\omega)\mathrm{e}^{\mathrm{i}\theta t}\mathrm{d}\theta=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\sqrt{S_{uu}(\omega)}A(\theta-\omega)\mathrm{e}^{\mathrm{i}\theta t}\mathrm{d}\theta \tag{15}$$

In Eq. (14) and Eq. (15), $A(\theta)$ is the Fourier integral transform of modulation function $a(t)$, given by

$$A(\theta)=\int_{-\infty}^{+\infty}a(t)\mathrm{e}^{-\mathrm{i}\theta t}\mathrm{d}t \tag{16}$$

Using Eq. (14) and Eq. (15), Eq. (12) can be developed further as

$$\widetilde{\mathbf{\Theta}}(\omega,t)=\int_{-\infty}^{+\infty}\mathbf{h}(\tau)\mathbf{e}\left(\frac{1}{2\pi}\int_{-\infty}^{+\infty}\tilde{X}(\theta,\omega)\mathrm{e}^{\mathrm{i}\theta(t-\tau)}\mathrm{d}\theta\right)\mathrm{d}\tau=\mathbf{g}(\omega,t)\sqrt{S_{uu}(\omega)}\mathrm{e}^{\mathrm{i}\omega t} \tag{17}$$

where

$$\mathbf{g}(\omega,t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathbf{H}(\theta+\omega)\mathbf{e}A(\theta)\mathrm{e}^{\mathrm{i}\theta t}\mathrm{d}\theta \tag{18}$$

Ref. [40] defined the evolutionary frequency response matrix $\mathbf{H}_{\mathbf{z}}(\omega,t)$ (using the notation in Ref. [40]), which was derived from the complex modal theory. $\mathbf{H}_{\mathbf{z}}(\omega,t)$ was expressed as the convolution integral of the impulse response function and the evolutionary amplitude harmonic loads, and the Runge-Kutta method with variable time steps was proposed to solve $\mathbf{H}_{\mathbf{z}}(\omega,t)$ in the time domain. Unlike $\mathbf{H}_{\mathbf{z}}(\omega,t)$, the amplitude modulation vector $\mathbf{g}(\omega,t)$ of response, which is defined in this paper, is determined by the modulation function of the random input excitation, and the solution process does not need time domain integration.

After obtaining $\mathbf{g}(\omega,t)$, the evolution power spectrum and time dependent variance of the structural random responses can be analyzed further. Taking $t_1=t_2=t$ in Eq. (11) and using Eq. (17), the time-dependent auto covariance matrix of responses $\mathbf{y}(t)$ is given by

$$\mathbf{R}_{yy}(t,t)=\int_{-\infty}^{+\infty}\widetilde{\mathbf{\Theta}}^{*}(\omega,t)\widetilde{\mathbf{\Theta}}(\omega,t)\mathrm{d}\omega=\int_{-\infty}^{+\infty}\mathbf{g}(\omega,t)\mathbf{g}^{\mathrm{T}}(\omega,t)S_{uu}(\omega)\mathrm{d}\omega \tag{19}$$

The integrand in Eq. (19) is the evolutionary power spectral density matrix of responses $\mathbf{y}(t)$ as

$$\mathbf{S}_{yy}(\omega,t)=\widetilde{\mathbf{\Theta}}^{*}(\omega,t)\widetilde{\mathbf{\Theta}}^{\mathrm{T}}(\omega,t)=\mathbf{g}(\omega,t)\mathbf{g}^{\mathrm{T}}(\omega,t)S_{uu}(\omega) \tag{20}$$

The evolutionary power spectra of the responses are derived using two-sided spectra in this section. Using the symmetry of the two-sided spectra, the time-dependent variance $\sigma_{\gamma}(t)$ of any response γ of a structure can be computed by

$$\sigma_{y}(t) = \int_{-\infty}^{+\infty} S_{yy}(\omega, t)\mathrm{d}\omega = 2\int_{0}^{+\infty} S_{yy}(\omega, t)\mathrm{d}\omega \tag{21}$$

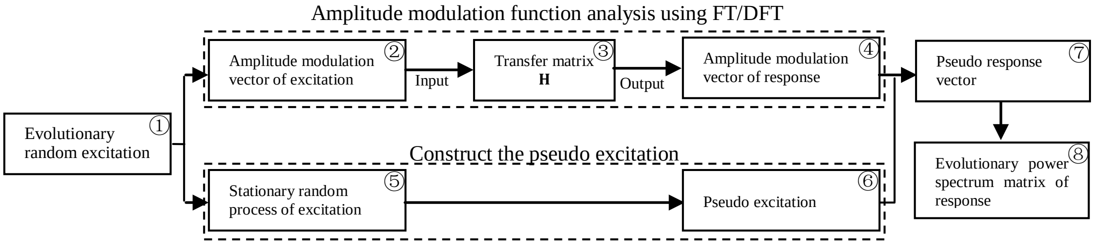


Fig.1. Block diagram of frequency domain analysis method of nonstationary random vibration

Based on the above derivation process(Eq. 14 - Eq. 20), the block diagram of the frequency domain analysis procedure of nonstationary random vibration using the combined pseudo-excitation and FT/DFT method is given (Fig.1). It can be seen from the block diagram that the evolutionary random excitation is divided into deterministic amplitude modulation part $\mathbf{e}a(t)$ and stationary random part $u(t)$. The amplitude modulation function analysis is implemented using FT/DFT (blocks ②→③→④ in Fig.1) as: the amplitude modulation vector of excitation $\mathbf{e}a(t)$ is transformed to $\mathbf{e}A(\theta)$ in the frequency domain; and the amplitude modulation vector of response $\mathbf{g}(\omega, t)$ is calculated using the frequency response function of the system (Eq. 18). Then, the constructed pseudo-excitation $\sqrt{S_{uu}(\omega)}\mathrm{e}^{\mathrm{i}\omega t}$ (block ⑤→⑥ in Fig.1) is combined with amplitude modulation vector of response $\mathbf{g}(\omega, t)$ (block ④ in Fig.1) into a pseudo-response vector $\widetilde{\mathbf{\Theta}}(\omega, t)$ (block ⑦ in Fig.1). Finally, evolutionary power spectral density matrix $\mathbf{S}_{yy}(\omega, t)$ is obtained using Eq. (20).

It can be concluded from Eq. (20) that: the random response possesses a mathematical form of an evolutionary nonstationary random process, whose modulation function is $\mathbf{g}(\omega, t)$. This modulation function of random output can be determined directly from the modulation function of the random input. It should be noted that the initial displacement and initial velocity are taken as zero in the above derivation. Eq. (20) is the relationship between the random input and the random output in the frequency domain, which is derived by the pseudo excitation method combined with the Fourier analysis.

In Refs. [41-42], the nonstationary random vibration input-output relationship was studied using the evolutionary amplitude matrix. A matrix differential equation was derived, which governed the relation between $\boldsymbol{\alpha}(\omega, t)$ (the evolutionary amplitude matrix of the input) and $\boldsymbol{\beta}(\omega, t)$ (the evolutionary amplitude matrix of the output). Based on the state-space formulation, the time domain integration method was established and the integration scheme for $\boldsymbol{\beta}(\omega,t)$ was given, when $\boldsymbol{\alpha}(\omega,t)$ was constant or varied linearly within integration step. Finally, the power spectrum of the output was calculated according to $\mathbf{S}_{YY}(\omega,t)=\boldsymbol{\beta}(\omega,t)\boldsymbol{\beta}^{*}(\omega,t)$. Note that $\boldsymbol{\alpha}(\omega,t)$, $\boldsymbol{\beta}(\omega,t)$ and $\mathbf{S}_{YY}(\omega,t)$ belong to the notation in Ref. [41]. Compared with the evolution spectrum formula in Ref. [41], the formula (Eq. (20)) established in this paper is a vector multiplication of the pseudo response $\widetilde{\boldsymbol{\Theta}}(\omega,t)$ and the matrix multiplication is not needed. For the pseudo response vector $\widetilde{\boldsymbol{\Theta}}(\omega,t)$, this paper also presents a different analytical strategy and numerical method from those in Refs. [41-42].

The proposed method is different from the analytical strategy based on the traditional frequency-time concept and as such the nonstationary random vibration analysis is performed completely in the frequency domain. The PEM-FT method is suitable for not only a nonstationary random vibration analysis with different types of modulation functions, but also a linear system with different types of damping. Assuming that the cross-power spectral matrix of the multidimensional random input is known, the method presented in this section can be readily generalized to multicorrelated nonstationary random processes.

3.2 PEM-DFT method of nonstationary random vibration analysis

In section 3.1, Fig.1 shows that when nonstationary random vibration is estimated by the proposed method, only amplitude modulation analysis of the input-output evolution spectrum is needed. The amplitude modulation analysis is a deterministic analysis, which can be implemented by discrete Fourier transform (DFT).

The amplitude modulation function analysis of the input-output is performed by the DFT, as follows: (1) The amplitude modulation function $a(t)$ (a slowly varying function) is discretized in the time domain and transformed into the frequency domain $A(\theta)$ by the DFT; (2) The expression of the frequency response function of the linear system is derived; (3) $\mathbf{G}(\omega,\theta)$ is calculated in the frequency domain and the response amplitude modulation vector $\mathbf{g}(\omega,t)$ is obtained by inverse discrete Fourier transform (IDFT).

The modulation function $a(t)$ is discretized at N sample points at a regular sampling time step $\Delta t=T/N$ within interval $[0,T]$ in the time domain as: $a(t=t_0=0)$, $a(t=t_1=$

$\Delta t), \cdots, a(t = t_n = n\Delta t), \cdots, a(t = t_{N-1} = T)$, denoted as $a_0, a_1, \cdots, a_n, \cdots, a_{N-1}$. According to DFT a_n at the nth discrete point can be expressed as a linear combination of N complex harmonic components [43]

$$a_n = \sum_{l=0}^{N-1} A_l \mathrm{e}^{\mathrm{i}\theta_l t_n} = \sum_{l=0}^{N-1} A_l \mathrm{e}^{\mathrm{i}(2\pi n l/N)} \tag{22}$$

where $\theta_l = l\Delta\theta$ is the lth circular frequency of the complex harmonic component:

$$\Delta\theta = 2\pi/T = 2\pi/(N\Delta t) \tag{23}$$

The amplitude and phase of the lth complex harmonic component are defined by complex coefficient A_l in Eq. (22), given below in [44]

$$A_l = \frac{1}{T}\sum_{n=0}^{N-1} a_n \mathrm{e}^{-\mathrm{i}\theta_l t_n}\,\Delta t = \frac{1}{N}\sum_{n=0}^{N-1} a_n \mathrm{e}^{-\mathrm{i}(2\pi n l/N)} \tag{24}$$

where the series expansion of A_l is DFT of the modulation function $a(t)$ of the nonstationary random excitation.

In Eq. (22) and Eq. (24) only the positive frequencies are considered. If a single-sided expansion is adopted, the A_l at the other side of $\theta_{N/2}$ must be complex conjugate with the corresponding side as

$$A_l = A_{N-l}^{*} \text{ for } \frac{N}{2} < l \le N-1 \tag{25}$$

It should be said that the frequencies at $N/2 < l \le N-1$ do not have a physical meaning, and this part is said to be associated with negative frequencies, when double sided Fourier series is adopted. So $\theta_{N/2}$ is defined as the highest frequency of the participatingharmonic components, which also can be represented by $\theta_{\max} = \frac{N}{2}\Delta\theta = \frac{\pi}{\Delta t}$.

The ordinary differential equation governing responses of a dynamic system also can be transformed into a set of algebraic equations through DFT. For the nonstationary random vibration analysis of a system the amplitude modulation vector $\tilde{\mathbf{A}}_l$ of responses can be given by

$$\tilde{\mathbf{A}}_l = \mathbf{H}_l \mathbf{e} A_l \quad (0 \le l \le N-1) \tag{26}$$

where $\mathbf{H}_l$ is equal to $\mathbf{H}(\theta = \theta_l + \omega)$.

For the single-sided Fourier expansion, $\mathbf{H}_l$ must be a complex conjugate of $\mathbf{H}_{N-l}$, when $l > N/2$. In order to satisfy this relationship, θ_l of $\mathbf{H}_l$ in Eq. (26) should take the following value:

$$\left.\begin{array}{r} \theta_l = l\Delta\theta \quad \text{when} \quad 0 \le l \le N/2 \\ \theta_l = -(N-l)\Delta\theta \quad \text{when} \ N/2 < l \le N-1 \end{array}\right\} \tag{27}$$

Further, $\mathbf{g}_n$ of the system at frequency ω and discrete time instant $t = t_n$, can be computed by

$$\mathbf{g}_n = \sum_{l=0}^{N-1} \widetilde{\mathbf{A}}_l \mathrm{e}^{\mathrm{i}\theta_l t_n} = \sum_{l=0}^{N-1} \widetilde{\mathbf{A}}_l \mathrm{e}^{\mathrm{i}(2\pi n l/N)} \tag{28}$$

Finally, the evolutionary power spectral density or the time-dependent variance of system responses can be computed from Eq. (20) or Eq. (21). It is worth noting that the deterministic process and the random process in the above derivation have been separated effectively, and as a result a high sampling frequency is not needed when complex coefficient A_l of modulation function $a(t)$ is computed. This is very beneficial in the nonstationary random vibration analysis. This advantage of the proposed method will be clearly demonstrated in the numerical examples in section 5.

4 Application of frequency domain method for nonstationary random vibration analysis

4.1 Nonstationary random vibration of systems with proportional damping

The equation of motion of a linear multi-degree-of-freedom structure subject to a ground acceleration excitation can be written as

$$\mathbf{M}\ddot{\mathbf{y}} + \mathbf{C}\dot{\mathbf{y}} + \mathbf{K}\mathbf{y} = -\mathbf{M}\mathbf{e}x(t) \tag{29}$$

where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are *n*-dimensional mass, stiffness and damping matrices, respectively; $\mathbf{e}$ is the index vector of excitation, $x(t)$ is a uniformly modulated nonstationary random acceleration input given by Eq. (5).

For a complicated engineering structures with many degrees-of-freedom, mode superposition method is usually used. Assuming that the first q $(q \ll n)$ eigenvalues ω_j^2 and mass-normalised eigenvectors $\boldsymbol{\varphi}_j$ (j=1,2, $\cdots, q$) have already been obtained, let $\mathbf{y} = \boldsymbol{\Phi}\mathbf{z}$, Eq.(29) can be converted into the following into equation in modal coordinate vector $\mathbf{z}$ as

$$\ddot{\mathbf{z}} + \bar{\mathbf{C}}\dot{\mathbf{z}} + \boldsymbol{\Omega}^2\mathbf{z} = -\boldsymbol{\Phi}^{\mathrm{T}}\mathbf{M}\mathbf{e}(t)a(t)u(t) \tag{30}$$

where $\boldsymbol{\Omega}^2 = \mathrm{diag}[\omega_1^2, \omega_2^2, \cdots, \omega_q^2]$ is a diagonal matrix comprised of the first q eigenvalues; $\boldsymbol{\Phi} = \left[\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2, \cdots, \boldsymbol{\varphi}_q\right]$ is the corresponding eigenvector matrix; and

$$\bar{\mathbf{C}} = \boldsymbol{\Phi}^{\mathrm{T}}\mathbf{C}\boldsymbol{\Phi} \tag{31}$$

When proportional damping is assumed, $\bar{\mathbf{C}}$ also is a diagonal matrix. Then Eq. (31) can be decomposed into mutually independent equations of a single modal coordinate

$$\ddot{z}_j + 2\zeta_j\omega_j\dot{z}_j + \omega_j^2 z_j = -\gamma_j a(t)u(t) \quad (j = 1,2,\cdots,q) \tag{32}$$

where ζ_j is jth modal damping ratio, and γ_j is jth modal participation factor

$$\gamma_j = \boldsymbol{\varphi}_j^{\mathrm{T}} \mathbf{Me} \tag{33}$$

Using Duhamel integral the solution of Eq. (30) can be expressed as

$$z_j = -\gamma_j \int_{-\infty}^{\infty} h_j(\tau) a(t-\tau) u(t-\tau) \mathrm{d}\tau \quad (j = 1,2,\cdots,q) \tag{34}$$

where $h_j(t)$ is impulse response function of the jth mode.

Substituting Eq. (34) into $\mathbf{y} = \boldsymbol{\Phi}\mathbf{z}$ gives

$$\mathbf{y}(t) = -\sum_{j=1}^{n} \gamma_j \boldsymbol{\varphi}_j \int_{-\infty}^{\infty} h_j(\tau) a(t-\tau) u(t-\tau) \mathrm{d}\tau \tag{35}$$

The derivation of time-dependent auto-covariance matrix of responses $\mathbf{y}(t)$ is similar to the derivation of Eqs. (6)- (11) in section 3.1, given by

$$\mathbf{R}_{yy}(t_1, t_2) = \int_{-\infty}^{+\infty} \widetilde{\boldsymbol{\Theta}}_1^*(\omega, t_1) \widetilde{\boldsymbol{\Theta}}_1^{\mathrm{T}}(\omega, t_2) \mathrm{d}\omega \tag{36}$$

$\widetilde{\boldsymbol{\Theta}}_1(\omega, t)$ in Eq. (36) is given by

$$\widetilde{\boldsymbol{\Theta}}_1(\omega, t) = \sum_{j=1}^{q} \int_{-\infty}^{+\infty} h_j(\tau) \tilde{x}(t-\tau, \omega) \gamma_j \boldsymbol{\varphi}_j \, \mathrm{d}\tau \tag{37}$$

where $\tilde{x}(t, \omega)$ is pseudo excitation as in Eq. (13) and $\widetilde{\boldsymbol{\Theta}}_1(\omega, t)$ is thepseudo response of the system subjected to pseudo excitation $\tilde{x}(t, \omega)$.

Similar to Eq. (17), using Fourier transform the Eq. (37) can be further derived as

$$\begin{aligned} \widetilde{\boldsymbol{\Theta}}_1(\omega, t) &= \left(\sum_{j=1}^{q} \mathbf{g}_j(\omega, t)\right) \sqrt{S_{uu}(\omega)} \mathrm{e}^{\mathrm{i}\omega t} \\ \mathbf{g}_j(\omega, t) &= \gamma_j \boldsymbol{\varphi}_j \int_{-\infty}^{+\infty} H_j(\theta + \omega) A(\theta) \mathrm{e}^{\mathrm{i}\theta t} \mathrm{d}\theta \end{aligned} \tag{38}$$

where $A(\theta) = \int_{-\infty}^{+\infty} a(t) e^{-\mathrm{i}\theta t} \mathrm{d}t$ is the Fourier transformation of the modulation function $a(t)$.

Making $t_1 = t_2 = t$ in Eq. (36) and using Eq. (38), the time-dependent auto-covariance matrix of $\mathbf{y}(t)$ can be obtained as

$$\mathbf{R}_{yy}(t, t) = \int_{-\infty}^{+\infty} \left(\sum_{j=1}^{q} \mathbf{g}_j(\omega, t)\right) \left(\sum_{j=1}^{q} \mathbf{g}_j(\omega, t)\right)^{\mathrm{T}} S_{uu}(\omega) \mathrm{d}\omega \tag{39}$$

The integrand in Eq. (39) is simply EPSD matrix of $\boldsymbol{y}(t)$, as

$$\mathbf{S}_{yy}(\omega, t) = \left(\sum_{j=1}^{q} \mathbf{g}_j(\omega, t)\right) \left(\sum_{j=1}^{q} \mathbf{g}_j(\omega, t)\right)^{\mathrm{T}} S_{uu}(\omega) \tag{40}$$

In Eq. (40) all coupling terms of the participant modes are considered and it is also called Complete Quadratic Combination (CQC) method. It can be seen that for power spectrum analysis of nonstationary random vibration, the EPSD of system response can be calculated from Eq. (40), which requires $\mathbf{g}_j(\omega, t)$ that is obtained from Eq. (38). Here, Eqs. (39) - (40) together are the mathematical manifestation of the PEM-FT for the nonstationary random vibration analysis of the multi-degree-of-freedom systems.

4.2 Nonstationary random vibration of systems with non-viscous damping

In structural dynamics, the damping model is used to describe the vibration energy dissipation behavior of a system. The damping model can be divided into the viscous damping model and the non-viscous damping model. The viscous damping model was first proposed by Thomson and developed by Voight. For the viscous damping model, it is assumed that the damping force is proportional to the velocity and is an idealized case, which simplifies the solution of the dynamic differential equation. However, for ultra-light space structures or composite structures widely used in aerospace industry, the viscous damping model leads to an energy loss proportional to the structural natural frequency and does not match the actual energy loss, and thus can result in significant errors. For a more accurate characterization of energy dissipation of real structures, the non-viscous damping model has attracted much attention [45-49].

4.2.1 Complex-valued damping

For the energy loss caused by the internal friction of materials, Myklestad proposed a complex damping model that assumes that the phase angle of the strain always falls behind the phase angle μ of the stress due to the damping effect, where μ can be approximated by a hysteresis constant [47]. According to this assumption, the damping term in the equation of motion is taken to be $\mathrm{e}^{\mathrm{i}\mu}$ at times the stiffness term. For the multi-degree-of-freedom discrete system, the complex damping model is adopted, and its equation of motion becomes

$$\mathbf{M}\ddot{\mathbf{y}}(t) + \mathrm{e}^{\mathrm{i}\mu}\mathbf{K}\mathbf{y} = -\mathbf{M}\mathbf{e}x(t) \tag{41}$$

For the Eq. (41), it is easy to obtain its frequency domain form by Fourier transform, and its frequency response function matrix of the system can be written as

$$\mathbf{H}(\omega) = \left(\mathrm{e}^{\mathrm{i}\mu}\mathbf{K} - \omega^2\mathbf{M}\right)^{-1} \tag{42}$$

After obtaining the frequency response function matrix of the complex-valued damped system, the random vibration analysis under the nonstationary load can be carried out by the frequency domain method proposed in this paper according to the Eqs. (17) - (20).

4.2.2 Exponential damping

Based on the fact that the dissipation force depends on the velocity time history, Adhikari proposed a convolution damping model whose mathematical expression is the convolution of velocity and kernel function [48]

$$\mathbf{f}_d(t) = \int_0^t \mathbf{G}(t-\tau)\dot{\mathbf{y}}(\tau)\mathrm{d}\tau \tag{43}$$

$$\mathbf{G}(t) = \sum_{k=1}^{k_{\max}} \mathbf{C}_k \chi_k(t), t \geq 0 \tag{44}$$

where $\mathbf{f}_d(t)$ is the damping force, $\mathbf{G}(t)$ is the kernel function matrix, $\mathbf{C}_k$ is the damping coefficient matrix, and $\chi_k(t)$ is the damping function, in which $k_{\max}$ is the number of different exponential damping components. In general, the kernel function in Eq. (44) can be an exponential function

$$\chi(t) = \varepsilon \mathrm{e}^{-\varepsilon t} \tag{45}$$

in which ε is the relaxation factor. The non-viscous damping model given by equation (43) is also called the exponential damping model

For the multi-degree-of-freedom discrete system, the exponential damping model is adopted, and its equation of motion can be expressed as [48]

$$\mathbf{M}\ddot{\mathbf{y}}(t) + \sum_{k=1}^{k_{\max}} \mathbf{C}_k \mathbf{s}_k(t) + \mathbf{K}\mathbf{y}(t) = -\mathbf{M}\mathbf{e}x(t) \tag{46}$$

$$\dot{\mathbf{s}}_k(t) + \varepsilon_k \mathbf{s}_k(t) = \varepsilon_k \dot{\mathbf{y}}(t) \ \ (k = 1, 2, \cdots, k_{max}) \tag{47}$$

in which, $\mathbf{C}_k$ is the damping coefficient matrix.

Eqs. (46)-(47) of the system with exponential damping is expressed in the corresponding frequency domain by [46]

$$\left(\mathbf{K} + \sum_{k=1}^{k_{\max}} \mathrm{i}\omega \frac{\varepsilon_k}{\mathrm{i}\omega + \varepsilon_k} \mathbf{C}_k - \omega^2 \mathbf{M}\right) \mathbf{Y}(\omega) = \mathbf{F}(\omega) \tag{48}$$

The frequency response function matrix of the exponentially damped system can be further obtained from Eq. (48) as

$$\mathbf{H}(\omega) = \left(\mathbf{K} + \sum_{k=1}^{k_{\max}} \mathrm{i}\omega \frac{\varepsilon_k}{\mathrm{i}\omega + \varepsilon_k} \mathbf{C}_k - \omega^2 \mathbf{M}\right)^{-1} \tag{49}$$

It can be seen that for the system with exponential damping, the equation of motion in the frequency domain has a compact and explicit expression. As will be shown later in the paper the proposed frequency domain method can be applied directly to the analysis of nonstationary random vibration of a system with the exponential damping.

4.3 Nonstationary random vibration of a beam elastic foundation under a moving stationary random load

The dynamic behaviour of a structure under a moving load has always been an important topic for researchers and it has an extensive engineering background [50]. Moving loads are generally used to represent the dynamic interaction between vehicle and track structure. Generally, there is often randomness at a contact interface because of track irregularity and unevenness of supports. Although a moving load can be modelled as a stationary random process, the response of a structure under a moving load must be a nonstationary random process due to the movement of the load. In this section, the dynamic responses of an infinite long Euler beam resting on a Kelvin foundation subjected to a moving stationary random force are investigated (see Fig.2). Introducing a concept called 'equivalent modulation function', this class of moving load problems also can be solved conveniently using the proposed frequency domain method.

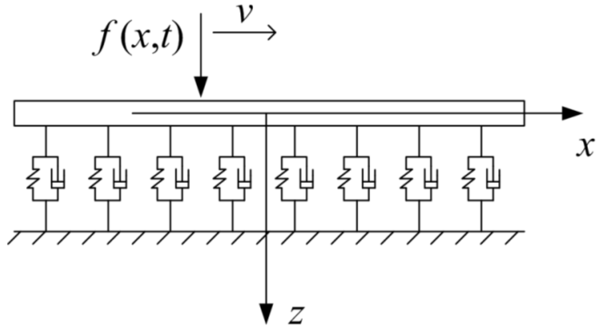


Fig.2. An Euler beam resting on a Kelvin foundation under a moving random load.

The equation of transverse motion of the beam can be described by

$$EI\frac{\partial^4 w}{\partial x^4} + Kw + \eta\frac{\partial w}{\partial t} + \rho\frac{\partial^2 w}{\partial t^2} = \delta(x - vt)f(t) \tag{50}$$

where $w = w(x,t)$ is the vertical displacement of the beam;EI and ρ are the flexural rigidity and mass per unit length of the beam, respectively;Kandη arethe foundation modulus and foundation damping; $f(t)$ is the force acting on the beam and it is taken as a stationary random excitation with velocity v; $\delta(\cdot)$ is Dirac function.Needless to say, the structural response of a structure under a moving stationary random force is nonstationary, due to the modulation function $a(t) \equiv \delta(x - vt)$ in Eq. (50).

For the steady response, the boundary conditions are given by

$$\lim_{x\to\pm\infty} w = 0,\ \lim_{x\to\pm\infty}\frac{\partial w}{\partial x} = 0 \tag{51}$$

Eq. (50) and Eq. (51) constitute the mathematical model governing the steady-state response of the infinite long Euler beam resting on the Kelvin foundation under a moving force.

The multi-domain Fourier transform of Eq. (50) in spatial variable $x$and time variable t, as in the wave number - frequency domain, is

$$(EIk_x^4 + K + \mathrm{i}\eta\theta - \rho\theta^2)Y(\theta,k_x) = F(\theta,k_x) \text{ or } Y(\theta,k_x) = H(\theta,k_x)F(\theta,k_x) \tag{52}$$

where

$$H(\theta,k_x) = (EIk_x^4 + K + \mathrm{i}\eta\theta - \rho\theta^2)^{-1}\mathrm{e}^{\mathrm{i}\theta t} \tag{53}$$

For the 'equivalent modulation function', being $a(t) \equiv \delta(x - vt)$ in this example, the pseudo excitation can be constructed and substituted into the right-hand side of Eq. (50) and this yields

$$EI\frac{\partial^4\tilde{w}}{\partial x^4} + K\tilde{w} + \eta\frac{\partial\tilde{w}}{\partial t} + \rho\frac{\partial^2\tilde{w}}{\partial t^2} = \delta(x - vt)\sqrt{S_{ff}(\omega)}\mathrm{e}^{\mathrm{i}\omega t} \tag{54}$$

The derivation process is similar to that for Eq. (17). The pseudo response $\tilde{w}(\omega,t)$ of the system under the pseudo excitation is given by

$$\begin{aligned}\tilde{w}(\omega,t,x) &= \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sqrt{S_{ff}(\omega)}\mathrm{e}^{\mathrm{i}\omega\tau}\delta(\zeta - v\tau)\cdot \\ &\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}H(\theta,k_x)\mathrm{e}^{\mathrm{i}\theta(t-\tau)}\mathrm{e}^{\mathrm{i}k_x(x-\zeta)}\mathrm{d}\,k_x\mathrm{d}\theta\right)\mathrm{d}\zeta\mathrm{d}\tau \\ &= g(\omega,t,x)\sqrt{S_{ff}(\omega)}\mathrm{e}^{\mathrm{i}\omega t}\end{aligned} \tag{55}$$

where

$$g(\omega,t,x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}H(\omega - k_x v, k_x)\mathrm{e}^{\mathrm{i}k_x(x-vt)}\mathrm{d}\,k_x \tag{56}$$

Eq. (55) is consistent with Eq. (17) and also has the same mathematical form as the evolutionary nonstationary random process whose modulation function is given in Eq. (56). Obviously, the response is nonstationary random due to the movement of the force. According to Eq. (20) and Eq. (21), the power spectrum of the nonstationary random vibration and time-dependent variance of the infinite long beam resting on the Kelvin foundation under the moving random force can be obtained.

5 Numerical examples

5.1 Example 1

A 21-bars plan truss is shown in Fig.3. The design parameters of the structure are: the length of horizontal bars and vertical bars is 5m; the product of the young's modulus and the cross-sectional areas of all bars is $EA = 3.0 \times 10^4$kN.

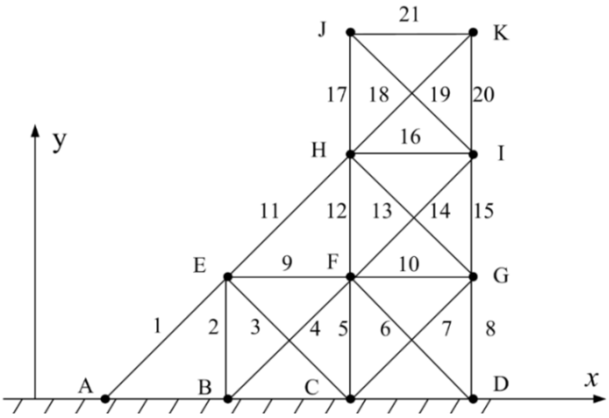


Fig.3. Plane truss.

The random vibration analysis of a structure under horizontal nonstationary random ground acceleration is investigated. The uniformity-modulated evolutionary random excitation is nonstationary random seismic acceleration, as defined in Eq. (5), and the auto-power spectrum of $u(t)$ follows the Kanai-Tajimi model, given by

$$S(\omega) = \frac{1 + 4\zeta_g^2(\omega/\omega_g)^2}{[1 - (\omega/\omega_g)^2]^2 + 4\zeta_g^2(\omega/\omega_g)^2} S_0 \tag{57}$$

where ω_g and ζ_g are the dominant frequency and the damping coefficient of the earthquake site, respectively; S_0 is the white noise intensity of the ground motion of the bed rock. In this example$\omega_g = 19.07\mathrm{s}^{-1}$, $\zeta_g = 0.544$, $S_0 = 142.75\ \mathrm{m}^{-2} \cdot \mathrm{s}^{-3}$. The random vibration analysis of this truss structure under the nonstationary seismic acceleration was performed by the frequency-time method introduced in [18, 32] and is now performed by the proposed frequency domain method so that a suitable comparison can be made.

5.1.1 Different modulation functions

Two different kinds of modulation functions are considered. These are [38]:

- three-segment piecewise modulation function:

$$a(t) = \begin{cases} I_0(t/t_1)^2 & 0 \le t \le t_1 \\ I_0 & t_1 \le t \le t_2 \\ I_0\exp[c_0(t - t_2)] & t \ge t_2 \end{cases} \tag{58}$$

- exponential modulation function:

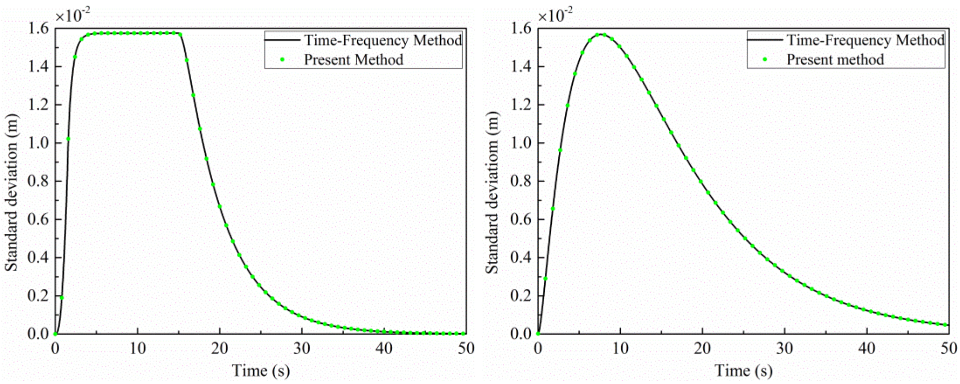
$$a(t) = \beta[\exp(-\alpha_1 t) - \exp(-\alpha_2 t)], \quad \alpha_1 < \alpha_2 \tag{59}$$

in which $I_0 = 1$, $t_1 = 1.5$, $t_2 = 15$, $c_0 = 0.2$; $\beta = 4$, $\alpha_1 = -0.0995$, $\alpha_2 = -0.199$.

In this calculation, the minimum and maximum frequencies of the analysis are $\omega_{\min} =$

0.2π rad/s and $\omega_{\max} = 40\pi$ rad/s, respectively; the frequency points are $\omega_k = \omega_{\min} + (\omega_{\max} - \omega_{\min}) \cdot \frac{k}{1000}$ $(k = 0, 1, 2, \cdots, 1000)$. The interval in the time domain analysis is $t \in [0,50\text{s}]$, the integration step is $\Delta t = 0.04$s. The sampling frequency and frequency points are respectively $f_s = 50$ Hz and $N = 2^{12}$ for the proposed frequency domain method. The modal damping ratio is taken as ζ=0.05.

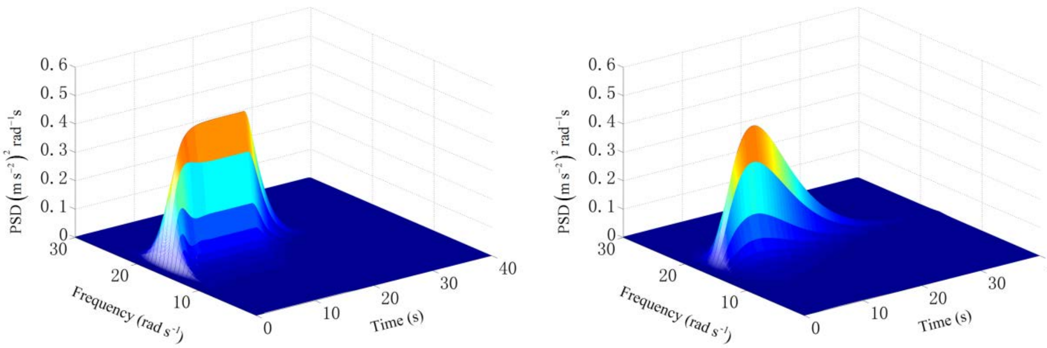
For comparison and verification, the nonstationary random vibration responses of the system are estimated using the traditional frequency-time methods and the proposed frequency domain method, respectively. The time-dependent standard deviation of the horizontal displacement response at node K of the truss is given inFig.4(a) and Fig.4(b), for the modulation functions given in Eqs. (58) and (59), respectively. From Fig.4(a) and Fig.4(b)it can be seen that the results obtained by the frequency domain method and traditional frequency-time method match exactly and this shows the validity of the proposed method. For the three-segment piecewise modulation function, the time-dependent standard deviation of the horizontal displacement response at node K illustrated in Fig.4(a) exhibits a nonstationary rise, a stationary steady state and a nonstationary falling. For the exponential modulation function, Fig.4(b) reveals that the time-dependent standard deviation curve will decline slowly after reaching the maximum. On computational efficiency: the computation time of the frequency-time method is 103.89s and the computation time of the proposed frequency method is 23.56s, for the three-segment piecewise modulation model; the computation time of the frequency-time method is 120.75s and the computation time of the proposed frequency method is 23.82s, for the exponential modulation model.



(a) (b)

Fig.4. Time-dependent standard deviation of the horizontal displacement response at node K:(a) for three-segment piecewise modulation function. (b) for exponential modulation function.

Fig.5(a) and Fig.5(b) show that the EPSD of the horizontal displacement responses at node K calculated by the proposed method for the three-segment piecewise modulation model and the exponential modulation model, respectively. From Fig.5(a) and Fig.5(b) it can be seen that, for both modulation models, the frequency interval of the peak sits within $\omega \in [10,20]\mathrm{rad/s}$, which is the result of the filtering characteristics of the system. The first natural frequency of the structure is known as ω_1 =16.93$\mathrm{rad/s}$. It is clear that the evolutionary process of the system responses can be revealed by the evolutionary power spectrum in the frequency-time domain, which can reflect accurately the vibration characteristics of the system itself and have a clear physical meaning, and provides an important reference value to the design and modification of a real structure.

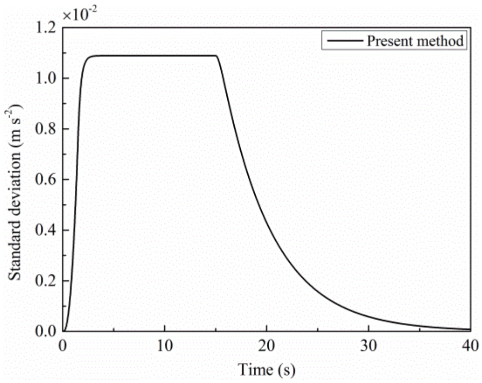


(a) (b)

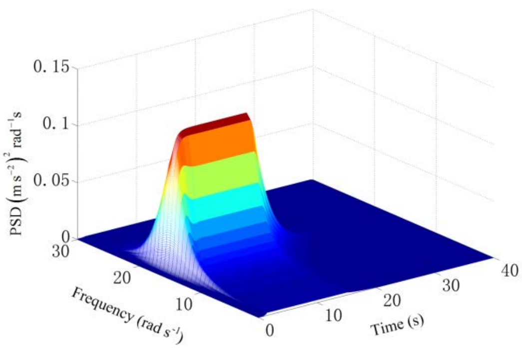
Fig.5. Evolutionary power spectrums of the horizontal displacement response at node K: (a) for three-segment piecewise modulation function. (b) for exponential modulation function.

5.1.2 Complex-valued stiffness matrix

Now a complex-valued stiffness matrix is considered as $\overline{\mathbf{K}} = \mathbf{K} * \exp(0.2\mathrm{i})$. All the other values remain the same as in section 5.1.1. The nonstationary random vibration analysis of the system is performed again. For the problem considered, instability in computation may appear when the step-by-step integration algorithm is applied directly to compute the responses of the system in the time domain. Some scholars believe that the main reason for this instability is that there exists an instability subset within the solution set of the equation of motion of the system with complex-valued stiffness matrix [26]. When the nonstationary random vibration of the system with complex-valued stiffness matrix is computed, the solution instability is likely to be encountered because the step-by-step integration needs to involve each frequency point using the frequency-time method. This problem can be avoided effectively using the proposed frequency domain method. There is now no need to convert the dynamic equation into the state space and Eqs. (17) - (20) can be solved directly. For the system with the complex stiffness matrix, Fig.6(a) and Fig.6(b) show that the time-dependent standard deviation and evolutionary power spectral density of the horizontal displacement response at node K, respectively. Clearly the behaviour is similar to that in section 5.1.1 — the time-dependent standard deviation also exhibits a nonstationary rise, a stationary steady stage and a nonstationary falling. This shows that the curve shape is determined by the nature of the curve characteristic of the modulation function. The resonant frequency region can be found intuitively from the evolutionary power spectrum given inFig.6(b). The computation time of the nonstationary random vibration analysis is 28.19s and the proposed method has excellent numerical stability.



(a)



(b)

Fig.6. The horizontal displacement response at node K when the complex stiffness matrix is used: (a) the time-dependent standard deviation. (b) the evolutionary power spectrum.

5.1.3 Different sampling frequencies

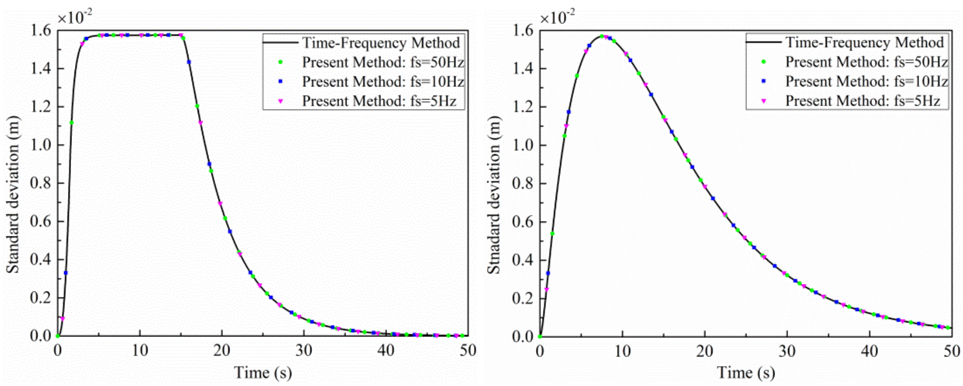
The theoretical development in section3 has demonstrated that the deterministic process and the random process of the nonstationary random excitation can be separated effectively. A high sampling frequency is not needed because the modulation function of the evolutionary nonstationary random excitation is a slowly varying function when the proposed frequency domain method is used for nonstationary random vibration analysis. To reach this conclusion, the following analysis cases are selected: (1)$f_s = 50\text{Hz}, N = 2^{12}$; (2)$f_s = 10\text{Hz}, N = 2^{10}$; (3)$f_s = 5\text{Hz}, N = 2^{9}$. In the analysis process, the same frequency interval as that in section 5.1.1, and the same modulation functions of the nonstationary excitation given by Eq. (58) and Eq. (59) are used.

The numerical results illustrated in Fig.7 and Fig.8 show the time-dependent standard deviation of the horizontal displacement response at node K for the three-segment piecewise modulation function and the exponential modulation function, respectively. For the convenience of comparative analysis, the results given by the frequency-time method are also displayed in Fig.7(a) and Fig.7(b). It can be seen that the results obtained at different sampling frequencies are identical to one another and also to the results from the frequency-time method.

The computation time for each case is listed in Tab. 1. When the sampling frequency $f_s = 5\text{Hz}$, the time durations of random vibration analysis are 3.55s and 3.01s for the two nonstationary random excitation, respectively. This shows the proposed frequency domain method has excellent efficiency. In addition, the evolutionary power spectrums of the horizontal displacement response at node K for the three-segment piecewise modulation model and exponential modulation model (at $f_s = 5\text{Hz}$) are shown in Fig.8(a) and Fig.8(b), respectively. They are identical to this same quantities obtained from $f_s = 50\text{Hz}$, shown in Fig.5(a) and Fig.5(b).

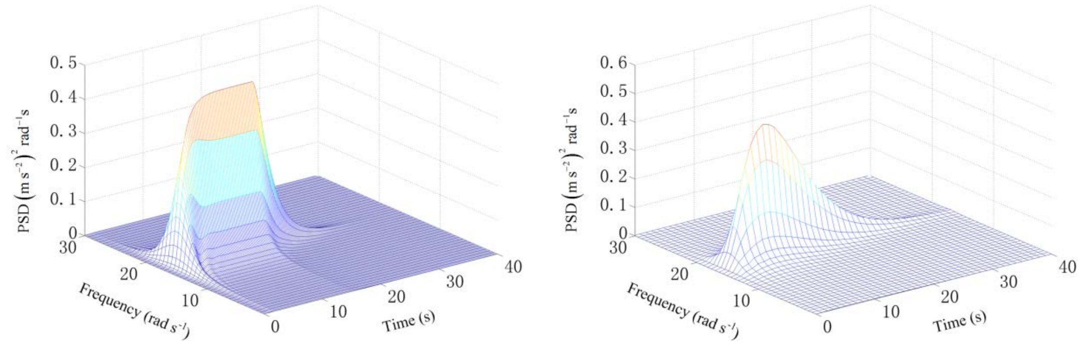
Table1. Computation time of nonstationary random analysis under different sampling frequency (s)

Three-segment piecewise modulation function				Exponential modulation function			
Frequency-time method	frequency domain method			Frequency-time method	frequency domain method		
	50Hz	10Hz	5Hz		50Hz	10Hz	5Hz
123.44	28.12	6.80	3.55	123.20	24.40	5.97	3.01



(a) (b)

Fig.7. The time-dependent standard deviation of the horizontal displacement response at node K at three different sampling frequencies: (a) for three-segment piecewise modulation function. (b) for exponential modulation function.



(a) (b)

Fig.8. The evolutionary power spectrum of the horizontal displacement response at node K at $f_s = 5$Hz: (a) for three-segment piecewise modulation function. (b) for exponential modulation function.

5.2 Example 2

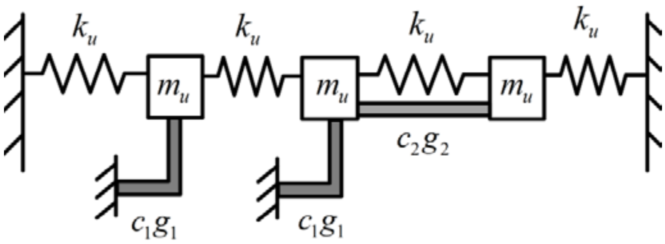


Fig.9. 3-DOF system with exponential damping

A3-DOF system with exponential damping studied in [27] is considered and illustrated below:

$$\mathbf{M} = \begin{bmatrix} m_u & 0 & 0 \\ 0 & m_u & 0 \\ 0 & 0 & m_u \end{bmatrix}, \mathbf{K} = \begin{bmatrix} 2k_u & -k_u & 0 \\ -k_u & 2k_u & -k_u \\ 0 & -k_u & 2k_u \end{bmatrix}$$
$$\mathbf{C}_1 = \begin{bmatrix} c_1 & 0 & 0 \\ 0 & c_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \mathbf{C}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & c_2 & -c_2 \\ 0 & -c_2 & c_2 \end{bmatrix} \tag{60}$$

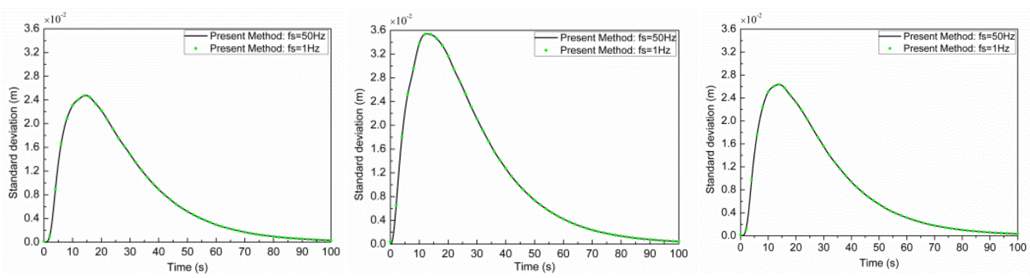
where $m_{\mathrm{u}} = 3.0\mathrm{kg}$, $k_{\mathrm{u}} = 2.0\mathrm{N/m}$, $c_1 = 0.6\mathrm{Ns/m}$, $c_2 = 0.2\mathrm{Ns/m}$.

In Fig.9 the shaded bars represent the non-viscous dampers with damping functions given by $\chi_i(t-\tau) = \varepsilon_i \mathrm{e}^{-\varepsilon_i(t-\tau)}$, $(i = 1, 2)$, $\varepsilon_1 = 1$, $\varepsilon_2 = 5$.

In this example, the second DoF is given an evolutionary nonstationary random excitation of $f_2(t) = 3a(t)u(t)$; the modulation function $a(t)$ is define in Eq. (59); $u(t)$ is a stationary random process whose power spectrum is defined in Eq. (57) and whose parameters are the same as in section 5.1. The nonstationary random vibration analysis of the system is performed at the same frequency interval as in section 5.1.1 and the sampling conditions are: (1)$f_s = 50\text{Hz}, N = 2^{13}$; (2)$f_s = 1\text{Hz}, N = 2^8$.

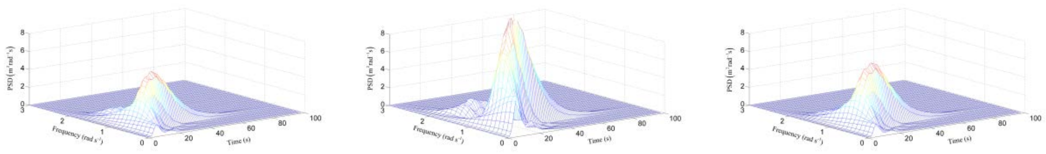
The time-dependent standard deviation of the first, second and third DoFs of the system are shown in Fig.10(a), Fig.10(b) and Fig.10(c), respectively. The results show that the curves of the time-dependent standard deviation have a similar shape for all DoFs of the system, in which the response of the second DoF is the highest and the response of the first DoF is the lowest. The responses at the first and third DoFs are asymmetric due to the asymmetric spatial distribution of the non-viscous damping in this structure. The evolutionary power spectrums of the first, second and third DoFs of the system are shown in Fig.11(a), Fig.11(b) and Fig.11(c), respectively. From these results, it can be seen that the evolutionary power spectrum distributions are concentrated around low frequencies due to the inherent filtering characteristics of the system. It should be stated that the three natural frequencies of the system are: ω_1 =0.6249 rad/s, ω_2 =1.154 rad/s and ω_3 =1.5087 rad/s. The results in Fig.11(a), Fig.11(b) and Fig.11(c) reveal that the non-viscous damping does not change the peak frequency range of the power spectrums, but rather the amplitude of the peak.

In addition, the computation time is 18.99s and 0.61s for $f_s = 50\text{Hz}$ and $f_s = 1\text{Hz}$, respectively. The results in Fig.10(a), Fig.10(b) and Fig.10(c) reveal that the frequency domain method also possesses high numerical accuracy when $f_s = 1\text{Hz}$. This is similar to section 5.1.2 where a step-by-step integration of the frequency-time method for nonstationary vibration analysis has been avoided. The analysis of a system with non-viscous damping is the same as the analysis of a system with exponential damping, and there is no need to compute a dynamic problem in state space, which is a distinct advantage.



(b) (b)(c)

Fig.10. Time-dependent standard deviation of the system: (a) first DoF; (b) second DoF; (c) third DoF.



(a) (b)(c)

Fig.11. Evolutionary power spectrum: (a) first DoF. (b) second DoF. (c) third DoF.

5.3 Example 3

In this section the nonstationary random vibration analysis of the infinite beam resting on the Kelvin foundation is investigated. The parameters of the structure are listed in Tab. 2, in which c_{cr} is the critical speed of a structure under determinate load.

Table2.Parameters of the system

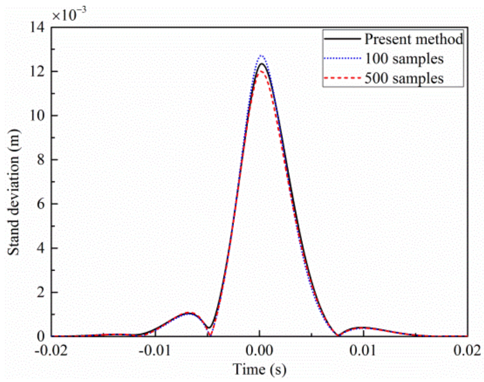
EI $(\mathrm{N}\cdot\mathrm{m}^2)$	ρ $(\mathrm{kg}\cdot\mathrm{m}^{-1})$	K $(\mathrm{N}\cdot\mathrm{m}^2)$	η $(\mathrm{kNs}\cdot\mathrm{m}^{-2})$	$c_{\mathrm{cr}}(\mathrm{m}\cdot\mathrm{s}^{-1})$
2.3×10^3	48.2	68.9×10^6	230	128.5

A band-limited white noise below is taken to represent the force's PSD as

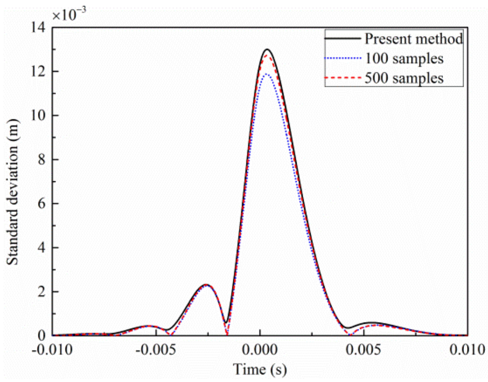
$$S_f(\omega) = 1\times10^8\mathrm{N}^2\mathrm{rad}\cdot\mathrm{s}^{-1}, \qquad \omega\in[0.1\pi, 100\pi] \tag{61}$$

Meanwhile, the response is also computed by the Monte Carlo method for the verification of the present method. From Fig.12 one can observe that the numerical results computed by the present method agree well with those computed by the Monte Carlo method. The random response of the vertical displacement at the origin is calculated both by the present method and by the Monte Carlo method with 100 and 500 samples. The results are compared in Fig.12 which gives the time-dependent standard deviation of the different velocity cases, from which it is seen that results calculated by the present method become close to those by the Monte Carlo method as the number of samples is increased. Numerical results show that the critical velocity of the structure under a moving random force ($100\ \mathrm{m\ s^{-1}}$) is a little lower than that under a moving constant force ($128.5\ \mathrm{m\ s^{-1}}$). When the random force is moving below the critical velocity ($100\ \mathrm{m\ s^{-1}}$), the standard deviation increases with growing velocity. The standard deviation achieves its maximum when the velocity is close to the critical velocity (Fig.12(b)), then it decreases with greater velocity in Figs.12(c-f).

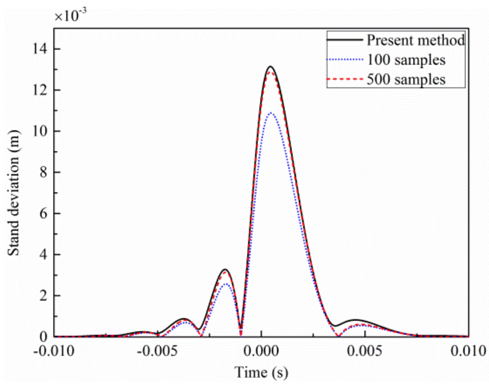
The responses can last long in low velocity ranges, and the response is somehow symmetrical with respect to $t=0$ (Fig.12(a)). As the velocity grows, the fluctuation in negative time is more obvious than that in positive time, and this phenomenon is particular obvious when the velocity is very large in Figs.12(e) and (f). One can also observe that the duration of the response becomes shorter, and the response in negative time decreases faster than that in positive time (Figs.12(c-f)). The reason for this phenomenon is due to the Doppler effect. When the force approaches the origin, the vibration frequency of the response grows and thus it causes the response to decrease faster due to the damping of the Kelvin foundation. But when the force is moving far away from the origin, the vibration frequency of the response decreases so it causes the response to last longer. The damping of the foundation causes a delay between the time when the force passes through the origin and the time when the standard deviation achieves its maximum. The delay vanishes when the damping of the foundation becomes zero. To achieve the results with acceptable accuracy, it costs about $2.3\ \mathrm{s}$ by the proposed frequency method. On the other hand, Monte Carlo method needs a large number of samples. To compute the response with 100 and 500 samples, it will take about $100\ \mathrm{s}$ and $877\ \mathrm{s}$, respectively. Therefore the proposed frequency domain method has a huge advantage over the Monte Carlo method, as no samples are needed by the present method.



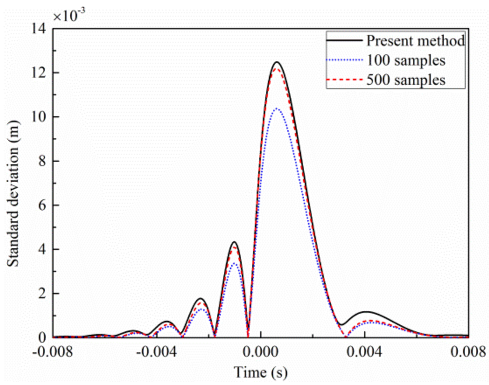
(a)



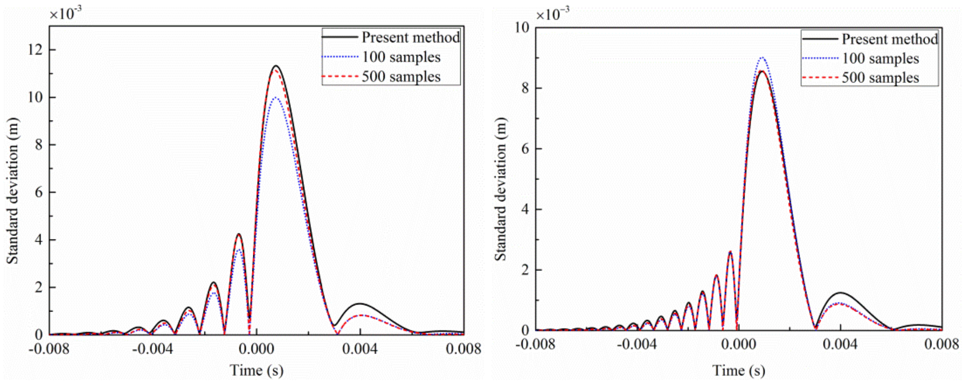
(b)



(c)



(d)



(e) (f)

Fig.12. Time-dependent standard deviation of vertical displacement at origin: (a)$v = 40\text{m s}^{-1}$. (b)$v = 80\text{m s}^{-1}$. (c)$v = 100\text{m s}^{-1}$. (d)$v = 128.5\text{m s}^{-1}$. (e)$v = 150\text{m s}^{-1}$. (f)$v = 200\text{m s}^{-1}$.

5.4 Example 4

Liao He Bridge is located between Panjin and Yingkou in Liaoning Province of China. It is a cable-stayed bridge whose main span length is 62.3+152.7+436+152.7+62.3=866m. The finite element model (Fig.13) of the bridge has 301 elements, 429 nodes and 1156 DoFs. The main deck and tower are modelled by three-dimensional beam elements with rigid arm and the cable is modelled by cable elements.

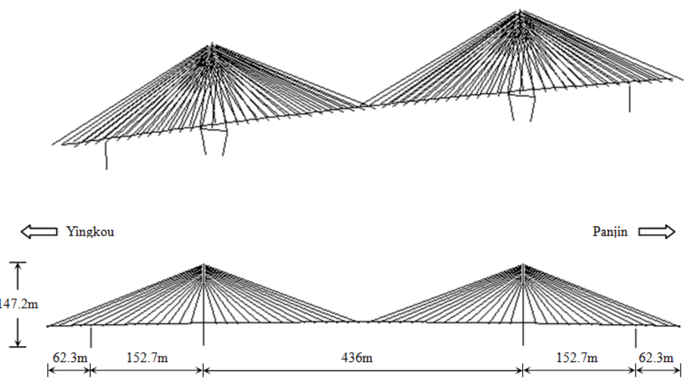


Fig.13. Finite element model of LiaoHe bridge.

The first 200 modes are used in model reduction; the frequency range of interest is $\omega \in [0.0,100]$rad/s, which corresponds to the range of periods of [0.046, 6.135]s; the analytical frequency step is $\Delta\omega = 0.2\ \text{rad/s}$.The damping ratio of participation modes is 0.05. The nonstationary excitation is considered an evolutionary random excitation with modulation function given by Eq. (58) in which $t_1 = 8.0\text{s}$, $t_2 = 20.0\text{s}$, $c = 0.2$ and the action time of the excitation is $[0,60\mathrm{s}]$. The acceleration spectrum of the ground surface's response is specified by the code in China (Code for Seismic Design of Building, GB50011-2001), whose intensity number is 7, site classification is 2, and earthquake classification is 1. The equivalent response spectrum of the site is transformed into the power spectrum of the stationary process by Kaul's method given in [51] as the input excitation of the bridge.

The peak of dynamic responses of a structure is most concerned in engineering design. After the power spectrum of the responses have been calculated, the probability distribution of the peak response can be obtained based on the random extreme value theory. In order to estimate the peak response under a nonstationary random excitation, the principle of the average energy equivalence between both processes mentioned above is adopted.

To obtain the power spectrum of a stationary random process, whose intensity is consistent with a nonstationary random process, the evolutionary power spectrum is averaged over time [32], as

$$\bar{S}_{yy}(\omega) = \frac{1}{\tau}\int_{\frac{t_1}{\sqrt{2}}}^{\frac{t_1}{\sqrt{2}}+\tau} S_{yy}(\omega,t)\mathrm{d}t \tag{62}$$

in which, $\tau = t_2 - \frac{t_1}{\sqrt{2}} + \frac{\ln 2}{c}$, t_1, t_2 and c are constants.

For any one possible response, denoted as y, the peak response of$y$is denoted by y_e, and the standard deviation of y is denoted by σ_y. The following dimensionless parameter is introduced as $\eta = y_e/\sigma_y$. Assume that any two crossing events, which are parts of a response due to random excitations that exceed a given limit, are independent of each other. The expectation and standard deviation of the peak response can be obtained from Ref. [52 - 53] as

$$E(\eta) = \sqrt{2\ln(\nu t_d)} + \frac{0.5772}{\sqrt{2\ln(\nu t_d)}} \tag{63}$$

$$\sigma_\eta = \frac{\pi}{\sqrt{6}}\frac{1}{\sqrt{2\ln(\nu t_d)}} \tag{64}$$

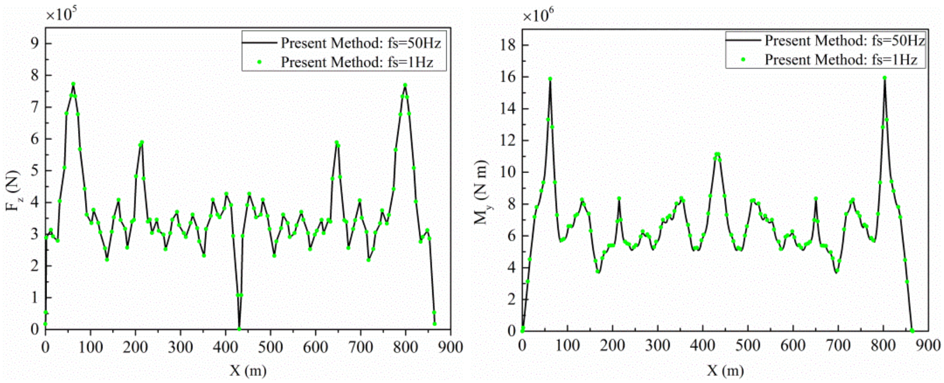
where ν is the mean zero-crossing rate of the process, and it can be written as

$$\nu = \frac{1}{\pi}\sqrt{\frac{\lambda_2}{\lambda_0}} \tag{65}$$

where λ_0 and λ_2 are the power spectral moments of the random process, and they are given by

$$\lambda_k = 2\int_0^\infty \omega^k \bar{S}_{yy}(\omega)\mathrm{d}\omega \ \ (k = 0, 2) \tag{66}$$

Firstly, the evolutionary power spectrum of the responses of the system under the nonstationary earthquake is computed using the frequency domain method proposed and it is transformed in to the equivalent stationary power spectrum based on the energy equivalent principle in Eq. (62).Then the peak responses are determined using Davenport’s method in Eq. (63). Considering SV wave (in the vertical plane)along the bridge, the peak responses of vertical shear force F_Z and horizontal bending moment M_y are computed using $f_s = 50\text{Hz}$ and $f_s = 1\text{Hz}$, respectively. The results obtained for both cases are identical (see Fig.14). Excluding the time taken for computing the free vibration response of structure, the computation time is 1680.39 s and 242.29 s, respectively. Fig.14(a) and Fig.14(b)reveal that the results of peak responses have a symmetric distribution due to the symmetry of the bridge and the excitation. In addition, the four larger values of the vertical shear force responses are situated at the positions of the piers and towers, respectively. The minimum responses are situated at the middle position of the main deck. The dynamic responses are dominated by the vertical modes of the bridge. Excluding the large change in the internal force at the supports, the dynamic responses of the main deck are in a substantially uniform state and this implies that design of the bridge’s main deck is excellent.



(a) (b)

Fig.14. The peak responses of the deck under the earthquake: (a) vertical shear force F_Z. (b) horizontal bending moment M_y.

For the seismic computations of long-span structures, various spatial effects of ground motion should be considered, including the wave-passage effect, the incoherence effect and the local site effect. The pseudo excitation method combined with precise integration method provides an effective means of analysis [54]. When considering the spatial effects of ground motion, the power spectral matrix of ground acceleration excitation is a Hermitian matrix. This multiple-input-multiple-output problem can be transformed into a linear superposition of 'equivalent single-source excitation problems' by properly decomposing the input power spectrum matrix. According to this strategy, the random vibration analysis of long-span structures considering the spatial effects of ground motion can still be performed according to the PEM / DFT method in this paper, which is also the author's intention to carry out further research.

6 Conclusions

For the nonstationary random vibration of structures caused by the evolutionary random excitation, the frequency-time analysis is generally executed using a step-by-step integration in the time domain, which is time consuming, and does not handle some situations very well, such as for non-viscously damped systems. In this paper, a frequency domain method for a nonstationary random vibration analysis is established by combining the pseudo-excitation method (PEM) improved by the authors and Fourier transforms (FT). The closed-form solution of the evolutionary power spectrum is derived by the proposed method and can be expressed explicitly as an evolutionary process modulated by a stationary random process. The main feature of the proposed method is that the deterministic process and the random process in the input/output are separated effectively. To determine the nonstationary random vibration, only the deterministic modulation function is needed. Additionally, when the discrete Fourier transform is used, a high sampling frequency is not needed because the modulation function of the nonstationary input is slowly varying in time. Several distinct numerical examples show that the proposed method is accurate and efficient.

Acknowledgments

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